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CAPITAL THEORY "PARADOX" AND PARADOXICAL RESULTS: RESOLVED OR CONTINUED?[1]

1. Introduction

Capital theory controversies and the associated with these "paradoxes" culminated in the decades o-neoclassical theory of value and distribution, and they showed that the wage-production price-profit rate curves can display shapes that are not consistent with the theoretical requirements of this theory. We do know that neoclassical prices are indexes of relative scarcities; thus, it is expected that, as the profit rate rises, corresponding to a fall in the wage rate, the prices of capital-intensive (labour-intensive) goods rise (fall).

The movement of relative prices of actual single-product economies as a result of changes in the distribution has been examined in a number of studies (Shaikh, 1984; 1998; 2014; Ochoa 1987; Tsoulfidis, 2008). These findings were extended both theoretically and empirically by formalizing the movement and the sign of the direction of production prices as an effect of changes in income distribution (Tsoulfidis, Mariolis, 2007; Mariolis, Tsoulfidis, 2009). In effect the analysis showed that once we normalize the production prices with the Sraffian Standard commodity (or industry), then the movement of production prices relative to labour values depends (regardless of the chosen numéraire) on the difference between the industry's vertically integrated capital-intensity evaluated in different prices from the standard ratio (the reciprocal of the maximum rate of profit), which remains invariable to changes in the distributive variables. This fundamental relationship is modified, to some extent, by the revaluation of the vertically integrated capital, this is Pasinetti's (1977) *price effect*. The input-output data of Greece (1970, 1988–1997), Japan (1970–2000) and China (1997), three quite different economies, spanning different time periods, showed that the *capital intensity effect* overshadows the price effect, and therefore determines both the size and the sign of the price movement and only in rare occasions and for unusually high or low (relative) rates of profit, the price effect may be strong enough as to change the direction and monotonic movement of relative prices. In these realistic but rare cases, we found that the path of estimated prices could have at most one extreme point making possible (in a fewer cases) the switching and ruling completely out the case of reswitching; thereby lending support to the idea of representing the estimated prices through a linear or, at most, a second order approximation as absolutely justifiable (Bienenfeld, 1988; Shaikh, 2011; Iliadi et al., 2014).

[1] The paper was presentation at the 19th ESHET Conference 14–16 May 2015, Roma Tre University. We thank the participants of this conference and especially Heinz Kurz, Fabio Petri, Bertram Schefold, and Persefoni Tsaliki for their comments and discussions. The usual caveat applies.

The claim that this paper raises is that we can further study this question and derive some meaningful theoretical results consistent with the available empirical evidence, as this is derived from the actual data of a number of diverse economies. More specifically, we start with a circulating capital single product economy, where wages are paid ex post and the analysis extends to include the case of fixed capital stock. The results in the case of circulating capital model, suggest a power law distribution of eigenvalues described **surprisingly well** by an exponential function, which is the same across the hitherto examined countries and also intertemporally. This is equivalent to saying that the module of eigenvalues contains only a few large values which typically fall abruptly to almost one-half of their maximal eigenvalue and then follows a long tail of very small eigenvalues that do not impart any noticeable influence on relative prices (Mariolis, Tsoulfidis, 2009; 2011; 2014). The results were much more pronounced in the case of fixed capital stock, where the power law distribution of eigenvalues resembles a nearly L-shaped pattern. In effect, the second eigenvalue is usually markedly smaller than the maximal eigenvalue, whereas the rest of eigenvalues are near zero and indistinguishable from each other, which is equivalent to saying that at most the second eigenvalue may exert some weak, rather negligible, influence on the relative prices. These findings prompted further investigation of the properties of the input-output structure of actual economies and the way in which this structure affects the movement of relative prices and the shape of the wage-profit rate curve. Our investigation reveals that the particular configuration of eigenvalues makes possible the mimicking of the motion of the entire economic system, at least, in terms of the production price-wage rate of profit trajectories, through the use of a single or just a few hypothetical sectors. These sectors bear meaningful similarities to what can be described by the Samuelson-Hicks-Spaventa 'corn-tractor' model (see Spaventa, 1970).

The UK input-output data on both circulating and fixed capital stock of the year 1990 represent an ideal testing ground for our theoretical findings. The results for the UK economy ascertain that (i) the non-monotonic production price-profit rate curves are relatively rare; and (ii) the wage-profit rate curves are quasi-linear. These findings imply that, although the real economies cannot be analyzed on the basis of 'neoclassical parables', the role of price effects in the revaluation of capital is of limited empirical significance (see, Krelle, 1977; Shaikh, 1984; 1998; 2012; Leontief, 1985; Ochoa, 1989; Petrović, 1987; Bienenfeld, 1988; Da Silva, Rosinger, 1992; Da Silva, 1993; Tsoulfidis, Maniatis, 2002; Angelousis, 2006; Han, Schefold, 2006; Iliadi et al., 2014). The remainder of the article is structured as follows. Section 2 deals with the theoretical issues lurking behind the movement of wage-production price-profit rate curves. Section 3 presents the empirical results. Section 4 argues that the results could be connected to the eigenvalue distribution of the system matrix. Section 5 estimates a 'corn-tractor' approximation that tends to contain the essential properties of the original system. Finally, Section 6 concludes.

2. The Mechanics of Price Adjustments

Consider a closed, linear system involving only single products, 'basic' commodities (in the sense of Sraffa, 1960, p. 7–8) and circulating capital. Assume that (i) the input-output coefficients are fixed; (ii) the system is strictly 'profitable' or, equivalently, 'viable', i.e. the Perron–Frobenius (P–F hereafter) eigenvalue of the irreducible $n \times n$ matrix of direct technical coefficients, $\mathbf{A}$, is less than 11,

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[1] The transpose of a $1 \times n$ vector $y \equiv [y_j]$ is denoted by $\mathbf{y}^{\mathrm{T}}$, and the diagonal matrix formed from the elements of $\mathbf{y}$ is denoted by $\hat{\mathbf{y}}$. Furthermore, $\mathbf{A}_j$ denotes the j th column of a semi-positive $n \times n$ matrix $\mathbf{A} \equiv [a_{ij}]$, $\lambda_{\mathbf{A}1}$ the P-F eigenvalue of $\mathbf{A}$ and $(\mathbf{x}^{\mathrm{T}}_{\mathbf{A}1}, \mathbf{y}_{\mathbf{A}1})$ the corresponding eigenvectors, while $\lambda_{\mathbf{A}k}$, $k=2,...,n$ and $|\lambda_{\mathbf{A}2}| \geq |\lambda_{\mathbf{A}3}| \geq ... \geq |\lambda_{\mathbf{A}n}|$, denote the non-dominant eigenvalues, and $(\mathbf{x}^{\mathrm{T}}_{\mathbf{A}k}, \mathbf{y}_{\mathbf{A}k})$ the corresponding eigenvectors. Finally, denotes the summation vector, i.e. $\mathbf{e} = [1,1,...,1]$, and $\mathbf{e}_j$ the jth unit vector.

and 'diagonalizable', i.e. **A** has a complete set of n linearly independent eigenvectors[1]; (iii) the net product is distributed to profits and wages that are paid at the end of the common production period; (iv) labour may be treated as homogeneous because relative wage rates are invariant; and (v) the profit rate, r, is uniform.

On the basis of these assumptions we can write

$$\mathbf{p} = w\mathbf{l} + (1+r)\,\mathbf{pA} \tag{1}$$

where **p** denotes a vector of production prices, w the money wage rate, and **l** (> **0**) the vector of direct labour coefficients. After rearrangement, equation (1) becomes

$$\mathbf{p} = w\mathbf{v} + r\mathbf{pH}$$

or

$$\mathbf{p} = w\mathbf{v} + \rho\mathbf{pJ} \tag{2}$$

or, if ρ > 1,

$$\mathbf{p} = w\mathbf{v}\,[\mathbf{I} + \rho\mathbf{J}]^{-1} \tag{2a}$$

where **I** denotes the $n \times n$ identity matrix, $v \equiv l[I - A]^{-1}$ the vector of 'vertically integrated' (Pasinetti, 1973) labour coefficients or 'labour values', and $\mathbf{H} \equiv \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$ the vertically integrated technical coefficients matrix, which is 'primitive'. Moreover, $\rho \equiv rR^{-1}$, $0 \leqslant \rho \leqslant 1$, denotes the 'relative (or normalized) profit rate', which equals the share of profits in the Sraffian Standard system (SSS), and $R \equiv \lambda_{A1}^{-1} - 1 = \lambda_{H1}^{-1}$ the maximum profit rate, i.e. the profit rate corresponding to $[w = 0, \mathbf{p} > \mathbf{0}]$, which equals the ratio of the net product to the means of production in the SSS (see Sraffa, 1960, chs. 4-5). Finally, $\mathbf{J} = R\mathbf{H}$ denotes the normalized vertically integrated technical coefficients matrix, $\lambda_{\mathbf{J}1} = R\lambda_{\mathbf{H}1} = 1$, and the moduli of the normalized eigenvalues of system (2) are less than those of system (1), i.e. $|\lambda_{\mathbf{J}k}| < |\lambda_{\mathbf{A}k}|\lambda_{\mathbf{A}1}^{-1}$ holds for all k (see, e.g. Mariolis, Tsoulfidis, 2014, pp. 213–214)[2].

If commodity $\mathbf{z}^{\mathrm{T}}$, with $\mathbf{vz}^{\mathrm{T}} = 1$, is chosen as the standard of value or illustrative numéraire, i.e. $\mathbf{pz}^{\mathrm{T}} = 1$, then equation (2a) implies

$$w = (\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1}\mathbf{z}^{\mathrm{T}})^{-1} \tag{3}$$

and

$$\mathbf{p} = (\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1}\mathbf{z}^{\mathrm{T}})^{-1}\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1} \tag{4}$$

Equation (3) gives a trade-off between w and ρ, known as the 'wage-relative profit rate curve' (WPC), and equation (4) gives the production prices as functions of ρ. It then follows that $w(0) = 1$, $\mathbf{p}(0) = \mathbf{v}$, $w(1) = 0$ and $\mathbf{p}(1) = \mathbf{y}_{\mathrm{J}1}$ (with $\mathbf{y}_{\mathrm{J}1}\mathbf{z}^{\mathrm{T}} = 1$). Finally, if Sraffa's Standard commodity (SSC) is chosen as the numeraire, i.e. $\mathbf{z}^{\mathrm{T}} = \mathbf{s}^{\mathrm{T}} \equiv [\mathbf{I} - \mathbf{A}]\mathbf{x}_{\mathbf{A}1}^{\mathrm{T}}$, with $\mathbf{lx}_{\mathbf{A}1}^{\mathrm{T}} = 1$, then

$$w = w^{\mathrm{S}} \equiv 1 - \rho \tag{5}$$

and

$$\mathbf{p} = (1 - \rho)\mathbf{v} + \rho\mathbf{pJ} \tag{6}$$

or

$$\mathbf{p} = (1 - \rho)\mathbf{v}[\mathbf{I} - \rho\mathbf{J}]^{-1} \tag{6a}$$

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[1] Given any **A** and an arbitrary $\varepsilon \neq 0$, it is possible to perturb the entries of **A** by an amount less than $|\varepsilon|$ so that the resulting matrix is diagonalizable (see, e.g. Aruka, 1991, p. 74–76).

[2] Within the well-known Leontief–Bródy treatment of fixed capital, **H** is replaced by $\mathbf{H}^{\mathrm{C}} \equiv \mathbf{A}^{\mathrm{C}}[\mathbf{I} - \mathbf{A}]^{-1}$, where $\mathbf{A}^{\mathrm{C}}$ denotes the matrix of capital stock coefficients.

When (i) the system is ‘regular’ (in the sense of Schefold, 1971, p. 11–23), i.e. $\mathbf{Ix}_{\mathbf{J}k}^{\mathrm{T}} \uparrow 0$ for all k (which is the empirically relevant case); and (ii) yJi, $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}}$ are normalized by setting $\mathbf{y}_{\mathbf{J}i}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}}=1$ and $\mathbf{v}[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}}=1$, then the WPC and the production price-profit rate relationships can be expressed in the following ‘spectral forms’ (Schefold, 2008, p. 14–20; Mariolis, Tsoulfidis, 2011, p. 91–92; Mariolis, 2015):

$$w=[(1-\rho)^{-1}d_1+\Lambda_k^w]^{-1},\ \Lambda_k^w \equiv \sum_{k=2}^{n}(1-\rho\lambda_{\mathbf{J}k})^{-1}d_k,\ \sum_{i=1}^{n}d_i=1,\ d_1=\mathbf{y}_{\mathbf{J}1}\mathbf{z}^{\mathrm{T}} \tag{7}$$

$$\mathbf{p}=w[(1-\rho)^{-1}\mathbf{y}_{\mathbf{J}1}+\Lambda_k^{\mathbf{p}}],\ \Lambda_k^{\mathbf{p}} \equiv \sum_{k=2}^{n}(1-\rho\lambda_{\mathbf{J}k})^{-1}\mathbf{y}_{\mathbf{J}k},\ \sum_{i=1}^{n}\mathbf{y}_{\mathbf{J}i}=\mathbf{p}(0) \tag{8}$$

where the terms Λ_k^w and $\Lambda_k^{\mathbf{p}}$ represent the effects of non-dominant eigenvalues on WPC and commodity prices, respectively, the d_i denote the coordinates of $\mathbf{z}^{\mathrm{T}}$ in terms of the right eigenbasis $[\mathbf{I}-\mathbf{A}]\mathbf{x}_{\mathbf{J}i}^{\mathrm{T}}$, and $\mathbf{p}(1)=d_1^{-1}\mathbf{y}_{\mathbf{J}1}$.

If there are strong quasi-linear dependencies amongst the technical conditions of production in all the vertically integrated industries, as we will see, this heuristic case is more geared towards reality than one at first sight might think, then it follows that $rank[\mathbf{J}]\approx 1$, or $|\lambda_{\mathrm{J}k}|\approx 0$ for all k, and $\mathbf{J}\approx\mathbf{J}^{\mathrm{A}}\equiv(\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}$. Thus, from equations (5) to (8) it follows that $\Lambda_k^w \approx 1-d_1$, $\Lambda_k^{\mathbf{p}} \approx \mathbf{p}(0)-\mathbf{y}_{\mathbf{J}1}$ and, therefore, both the WPC and the relative production price-profit rate relationships tend to be rational functions of degree 1 (homographic functions):

$$w\approx w^A \equiv [(1-\rho)^{-1}d_1+\sum_{k=2}^{n}d_k]=w^S[1+\rho(d_1-1)]^{-1} \tag{9}$$

$$p\approx p^A \equiv w^S[1+\rho(d_1-1)]^{-1}[(1-\rho)^{-1}d_1p(1)+\sum_{k=2}^{n}y_{Jk}]$$

or

$$p\approx p^A=[1+\rho(d_1-1)]^{-1}[p(0)+\rho(d_1p(1)-p(0))] \tag{10}$$

That is, the system tends to behave as a reducible two-industry economy without ‘self-reproducing non-basics’ (in the sense of Sraffa, 1960, Appendix B).

These approximations have the following properties:

(i). Their accuracy is directly related to the magnitudes of $|\lambda_{\mathbf{J}k}|^{-1}$.

(ii). They are exact at the extreme, economically significant, values of ρ.

(iii). When $rank[\mathbf{J}]=1$, i.e. $\mathbf{J}=(\mathbf{y}_{\mathbf{J}1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}})^{-1}\mathbf{x}_{\mathbf{J}1}^{\mathrm{T}}\mathbf{y}_{\mathbf{J}1}$, they become exact for all ρ. In that case, $\mathbf{J}$ can be transformed, via a semi-positive similarity matrix $\mathbf{T}$, into (Schur triangularization theorem; see, e.g. Meyer, 2001, p. 508-509)

$$\tilde{J}\equiv T^{-1}JT=\begin{bmatrix} 1 & \tilde{J}_{12} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{bmatrix} \tag{11}$$

where the first column of T is $\mathbf{x}_{J1}^{\mathrm{T}}$ (the remaining columns are arbitrary), and $\tilde{\mathbf{J}}_{12}$ is a $1\times(n-1)$ positive vector: if, for instance,

$$T=[x_{J1}^T,e_2^T,\ldots,e_n^T] \tag{11a}$$

then

$$\tilde{J}_{12}=(y_{J1}x_{J1}^T)^{-1}[y_{2J1},y_{3J1},\ldots,y_{nJ1}] \tag{11b}$$

That is, the original system is economically equivalent to an $n \times n$ 'corn-tractor' system, even if J is irreducible. Thus, the first row in the transformed matrix **J** represents our constructed industry which can be characterized as 'hyper-basic'[1].

(iv). If SSC is chosen as the numéraire, then $d_1 = 1$ and, therefore, approximation (10) becomes

$$\mathbf{p} \approx \mathbf{p}^A = \mathbf{p}(0) + \rho(\mathbf{p}(1) - \mathbf{p}(0)) \tag{12}$$

which coincides with Bienenfeld's (1988) linear approximation formula for the price vector. By contrast, from equations (8), $d_1 = 1$ and $d_k = 0$, it follows that

$$\dot{p} \equiv dp / d\rho = -\Lambda_k^p + (1-\rho)\dot{\Lambda}_k^p$$

which implies that in the *general* case the individual components of **p** can change in a complicated way as ρ changes.2

For the purposes of this paper it suffices to focus only on equation (6) from which we get

$$pv^{-1} = (1-\rho)e + \rho pJv^{-1}$$

or

$$pv^{-1} - e = \rho R(pHv^{-1} - R^{-1}e)$$

or, in terms of an industry j,

$$p_j v_j^{-1} - 1 = R\rho(k_j - R^{-1}) \tag{13}$$

where $k_j \equiv pH_j v_j^{-1}$ denotes the capital-intensity of the vertically integrated industry producing commodity j, and $R-1$ the capital-intensity of the SSS or Standard industry. The differentiation of equation (13) gives

$$\dot{p}_j v_j^{-1} = Rk_j(e_{kj} - D_j) \tag{14}$$

where $e_{kj} \equiv \dot{k}_j \rho k_j^{-1}$ denotes the elasticity of kj with respect to $\boldsymbol{\rho}$, and $D_j \equiv (Rk_j)^{-1} - 1$ the percentage deviation of the capital-intensity of the SSS from k_j (for an exploration of these relationships, see Mariolis, Tsoulfidis, 2009). The term e_{kj} represents the 'price effect' (indirect or Sraffian effect), which depends on the interindustry ramifications of the entire economic system and, therefore, is not predictable at the level of any single industry, while D_j represents the 'capital-intensity' effect (direct or traditional effect). From equation (14) it follows that the necessary condition for the violation of the traditional condition

$$D_j < (>)\ 0 \Leftrightarrow \dot{p}_j v_j^{-1} > (<)\ 0 \tag{15}$$

is the existence of a value of ρ, such that e_{kj} and D_j have the same sign, while the sufficient condition is $e_{kj} < (<)\ D_j < (>)\ 0$ or, alternatively,

$$e_{kj} D_j^{-1} = \dot{k}_j \rho (R^{-1} - k_j)^{-1} > 1 \tag{16}$$

Relation (16) signifies that the violation of the traditional condition is 'more unlikely':
(i) the smaller is the value of the relative profit rate; and/or

(ii) the greater is the difference between the capital-intensity of the SSS and the capital-intensity of the vertically integrated industry under consideration.

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[1] For other polar cases, which are also interesting, both theoretically and empirically, and include more than one hyper-basic industry, see (Mariolis, Tsoulfidis, 2014, p. 214–215, 2015, ch. 5).

[2] Nevertheless, C. Bidard, H. G. Ehrbar, U. Krause and I. Steedman have detected some 'monotonicity (theoretical) laws' for the relative prices (see Bidard, Ehrbar, 2007, and the references therein).

3. Results from the UK Economy

The results that follow correspond to equations (13) and (14), and are typical from a number of countries that have been tested (Mariolis, Tsoulfidis, 2015). We present those of the UK economy whose input-output data have not been used in similar experiments and, at the same time, includes both circulating and fixed capital stock; so we can compare the results and their probable differences[1].

3.1. Circulating Capital Model

The change in prices with respect to the relative profit rate is in most cases monotonic, as this is displayed in Fig. 1. There are also exceptions and these are displayed in a separate and enlarged graph in Fig. 2. We observe that all four curves in Fig. 2 display extreme points, and three from these curves cross the line of equality between production prices and labour values (‘price-labour value reversal’). Furthermore, despite the extrema character of the movement of these four curves, their deviation from the line of production price-labour value equality is less than 5 percent, a result that implies that the difference of capital-intensities of these four industries from that of the SSS is expected to be minimal. Thus, the ‘price effect’ may become particularly pronounced and to change the characterization of capital-intensities of the industries at hand. But let us see below the mirror image of the price movement, i.e. the change in the capital-intensities. The above results suggest that the specific trajectories of relative prices are dominated by the ‘capital-intensity’ effect (condition (15)).

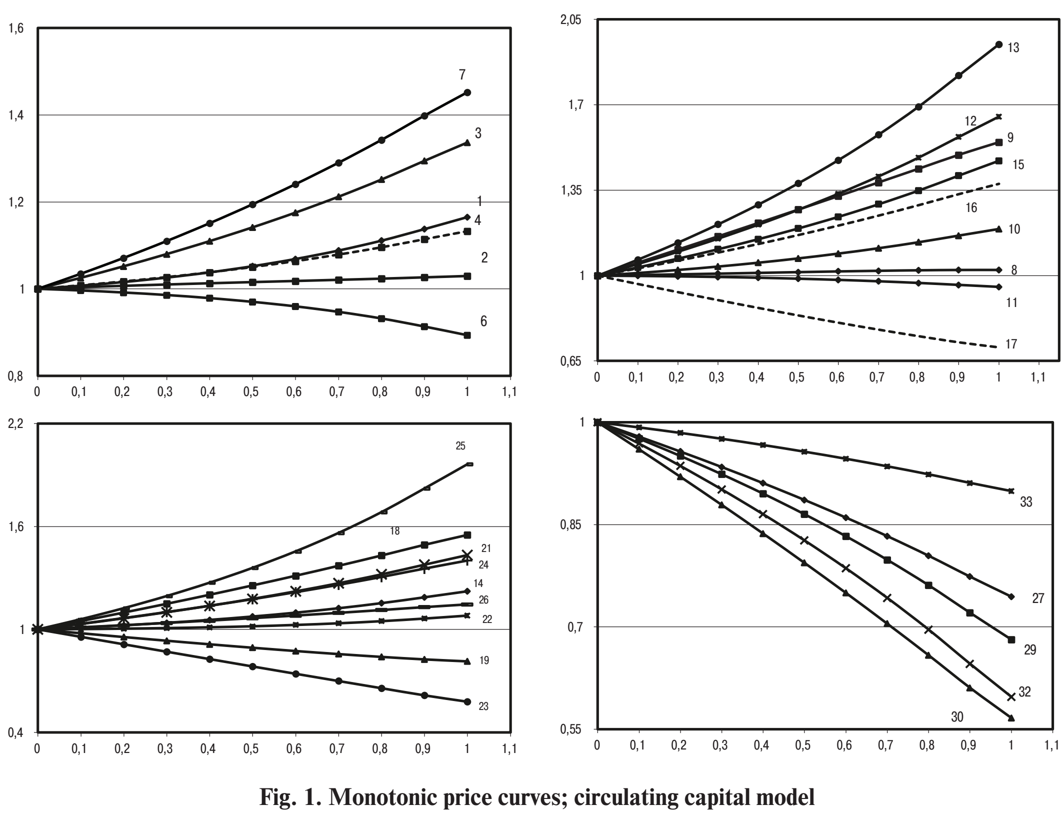


Fig. 1. Monotonic price curves; circulating capital model

[1] Each industry’s number can be identified in the Appendix.

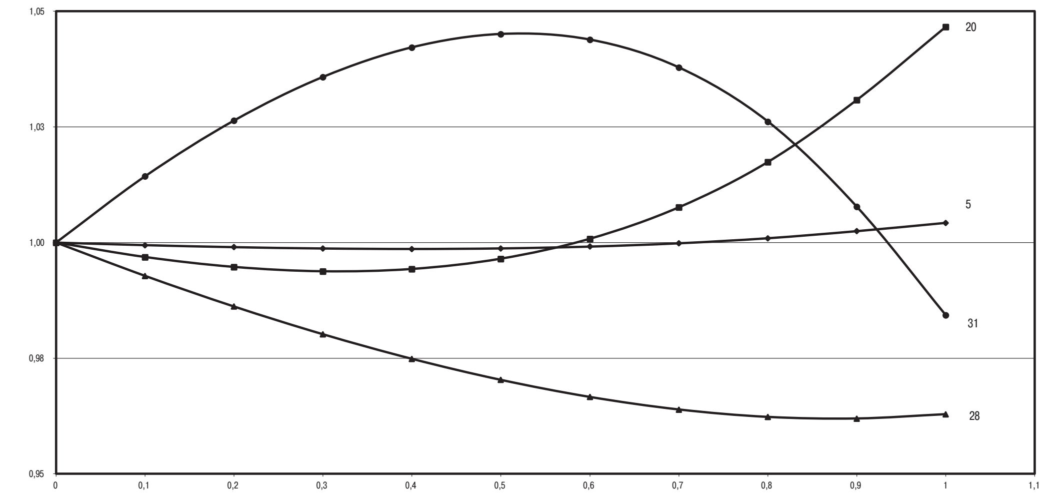


Fig. 2. Non-monotonic price curves; circulating capital model

In Fig. 3, below we display the capital-intensities of each of the 29 industries as a function of ρ, and we observe that the change in capital intensities is 'sluggish' and predictable in all cases.

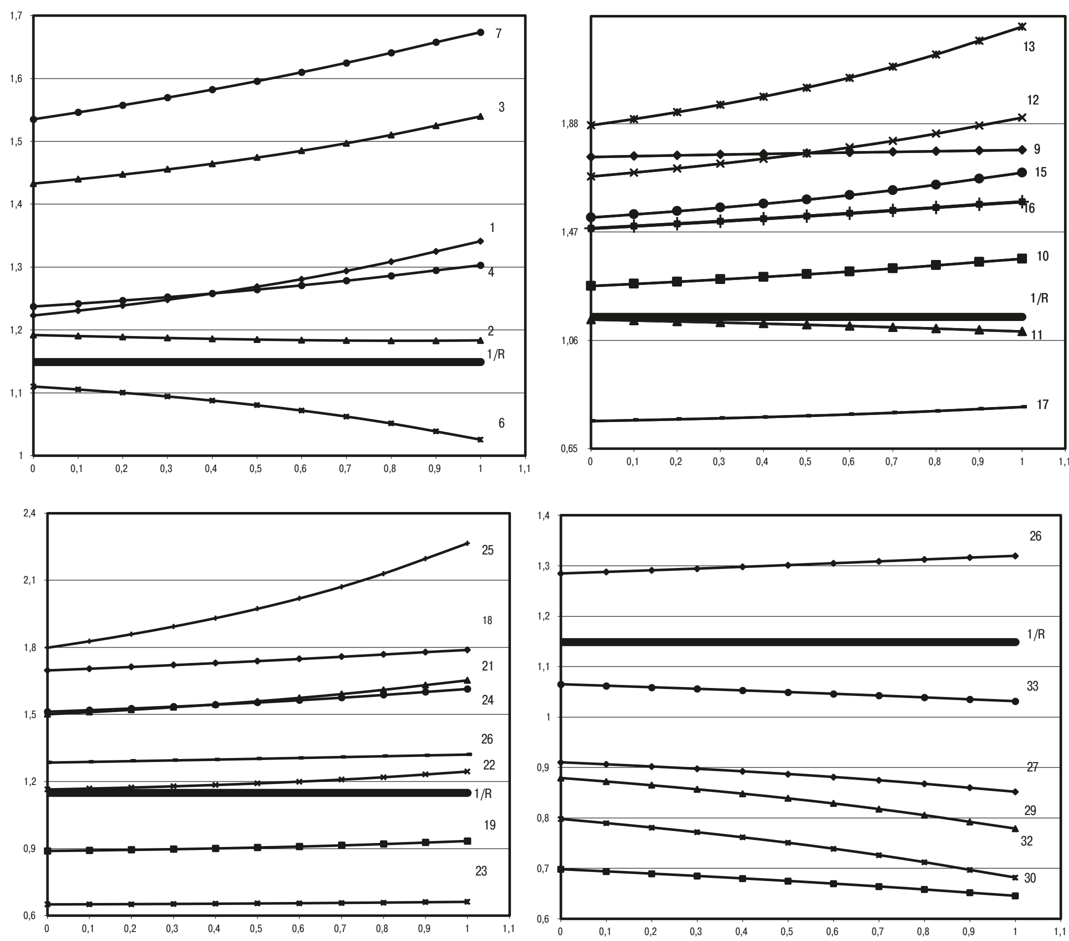


Fig. 3. Capital-intensities associated with the monotonic price curves, and the Standard capital-intensity; circulating capital model

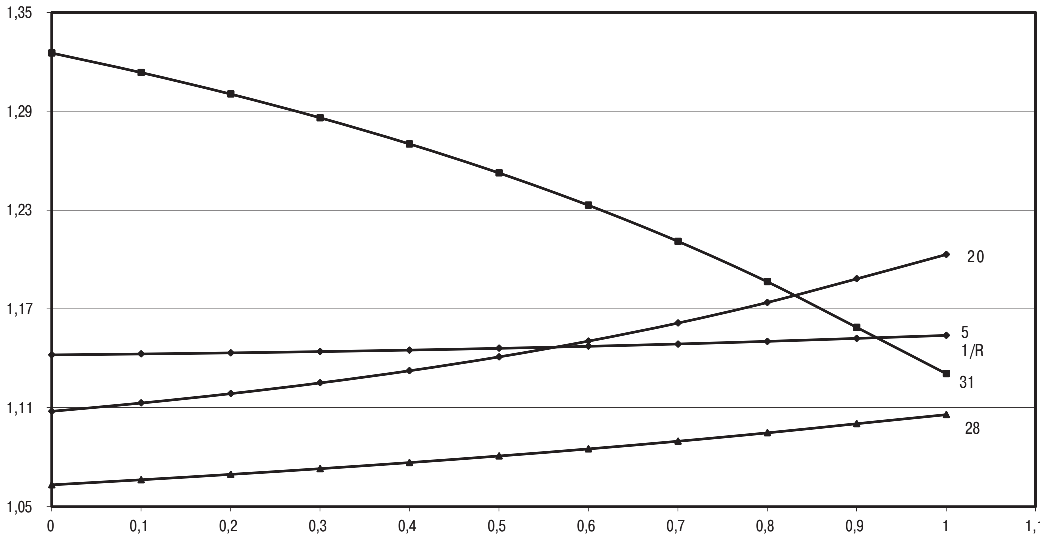


Fig. 4. Capital-intensities associated with the non-monotonic price curves, and the Standard capital-intensity; circulating capital model

A result which indicates that the change of characterization of an industry from capital-intensive to labour-intensive is a rather rare phenomenon and the probability of its occurrence increases when the capital-intensity of such an industry is near the Standard capital-intensity, $R^{-1} \cong 1{,}149$.

In Fig. 4 are displayed the trajectories of the four industries with the non-monotonic behaviour. For instance, industry 5 (Wood products & furniture) 'starts' as a labour intensive industry and ends up as a capital-intensive industry for a relative rate of profit higher than 0,6; the same is true for industry 20 (Other transport)[1]. At the minimum value of the relative profit rate, industry 31 (Finance and Insurance) is definitely a capital-intensive one and its initial difference from the Standard capital-intensity is by far higher than that of the other three industries. However, as the relative profit rate increases, the 'price effect' is strengthened giving rise to a maximum in the price-value trajectory, and then changes the characterization of this industry from capital-intensive to labour-intensive (see condition (16)). Finally, industry 28 (Restaurants and Hotels) maintains its character of a labour-intensive industry and the movement of its capital-intensity towards the Standard one gives rise to a minimum but not a reversal in the price-value trajectory.

3.2. Fixed Capital Model

The next experiment includes the case of fixed capital stock, where we examine the extent to which our findings in circulating capital model apply to this case. For this purpose, we repeat our experiment with the changes in distribution, as these are reflected in the new relative rate of profit and we estimate the new set of prices of production as the relative rate of profit increases from zero and approximates one. The trajectories of the production prices are displayed in the set of four graphs in Fig. 5.

[1] Industry 5 seems to possess at least one of the Ricardian properties of an invariable measure of value, that is, its capital-intensity remains approximately the same as income distribution changes. The same is true but to a lesser extent with industry 20.

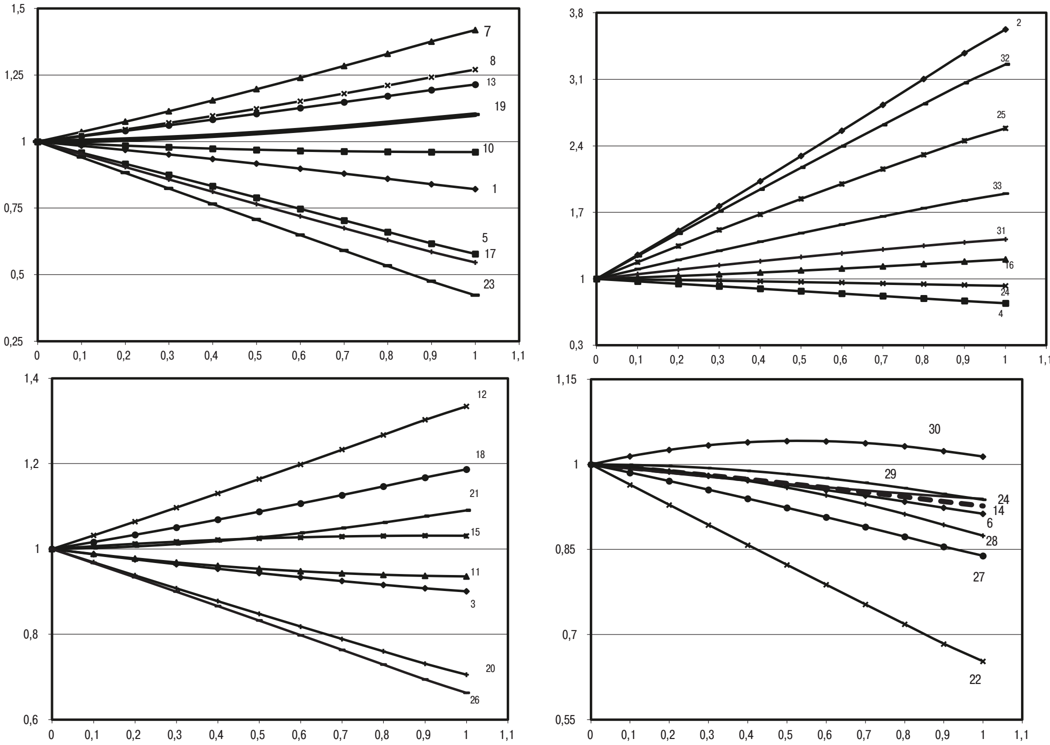


Fig. 5. Price curves; fixed capital model

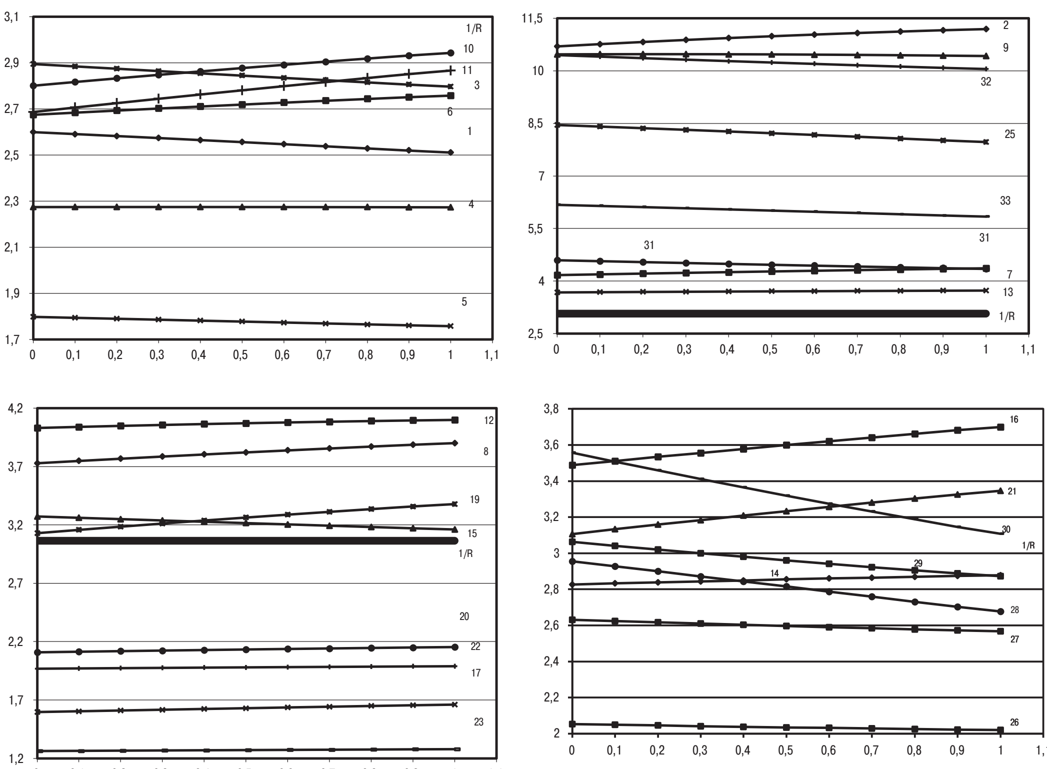


Fig. 6. Capital-intensities and the Standard capital-intensity; fixed capital model

We thus observe that almost all prices move monotonically. Only few industries display some short of curvature, but in no way changes in the direction of the movement of production prices, with the exception of industry 15 (Non-Electrical Machinery) and industry 30 (Communications), which display a maximum at $\rho \cong 0,9$ and $\rho \cong 0,5$, respectively. It is also observed in Fig. 6 that from all the capital-intensity curves that of industry 30 presents interest as it falls and approximates, but does not cross, the Standard capital-intensity line, $R^{-1} \cong 3,065$, and so the respective relative price although it displays a maximum nevertheless it does not cross the line of equality of production price and labour values. Similarly, industry 15 whose starting point is above but near the Standard capital-intensity and as the relative rate of profit rises the capital-intensity of industry 15 comes closer and closer to the Standard capital-intensity but no crossing is observed.

The position of the capital-intensity curves with respect to the Standard capital-intensity line is maintained in all cases. Most of them are nearly parallel to the latter line, while the differences $\left|k_j - R^{-1}\right|$ tend to be far larger than those of the circulating capital case. In effect, we found that, at $\rho = 0$, the capital-intensities in the circulating capital model, as these are estimated from $p(0)H_j v_j^{-1}$ gave an arithmetic mean equal to 1,25 and a standard deviation of 0,27 with a coefficient of variation of 0,21. The capital-intensities in the capital stock model, $p^C(0)H_j^C v_j^{-1}$, gave an arithmetic mean equal to 3,61 and a standard deviation of 1,72 which a coefficient of variation of 0,48, which is at least twice as high as that of the circulating capital model. These results make on an average much harder to find production price-labour value reversals in the capital stock model, while it seems that they relate to the fact that the matrix of capital stock coefficients, $\mathbf{A}^C$, contain many zeros or near-zero elements and, therefore, $\mathbf{H}^C$ is reducible without self-reproducing non-basics.

3.3. Wage-Profit Rate Curves

In order to complete the picture, we estimate the WPCs (equation (3)), measured in terms of the actual gross output vector, $\bar{\mathbf{x}}^T$, for both the circulating and the fixed capital cases. It is thus observed that the deviations of the WPCs from $w^S \equiv 1-\rho$ are not so large, indicating that the relevant price effects could be considered relatively weak. Therefore, for relatively low values of ρ, we can safely write (see equation (2))

$$w = 1 - \rho pJz^T \approx 1 - \rho p(0)Jz^T$$

where $z^T = (v\bar{x}^T)^{-1}\bar{x}^T$.

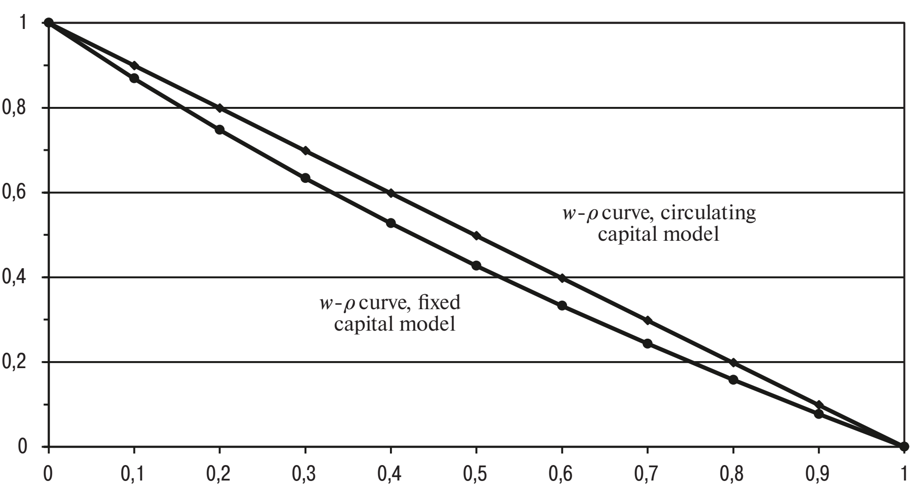


Fig. 7. The WPCs; circulating and fixed capital models

It is interesting to note that the WPC for the circulating capital model is almost a straight line, the maximum deviation from the straight line has been estimated at 0,21% and this occurs at a relative rate of profit of 60%. The WPC in the case of fixed capital model is curvilinear and its deviation from the straight line is higher than that of the circulating capital model, but not very different from the straight line, the maximum deviation is 7,9% and it occurs at a relative rate of profit of 50%.

4. Eigenvalue Distributions and Linear Approximation

The spectral forms of the wage-production price-profit rate relationships (see equations (7) and (8)) suggest that these empirical findings could be connected to the eigenvalue distributions of matrices **J**. Fig. 8 displays the moduli of those eigenvalues, and in the Appendix 3 we display the list of eigenvalues for both the circulating capital and the fixed capital model. It is interesting to note that the second eigenvalue of the fixed capital model is much lower than that of the circulating capital model. This finding has to do with the fact that the matrix of fixed capital coefficients contain industries that produce no capital goods and in this sense they are non-basic industries, whereas many of the industries that produce capital goods have many zero elements and therefore are not very far from being no basics. Furthermore, the matrix of fixed capital coefficients $\mathbf{A}^C$,because its non-zero elements are usually much higher than those of the matrix, imposes its form on its vertically integrated expression $\mathbf{A}^C[\mathbf{I} - \mathbf{A}]^{-1}$ and by doing so enhances its linear dependence measured by the so called spectral gap, that is, the difference between the first and the second eigenvalue. Fig. 9 and 10 display the actual price curves and their *linear* approximations (equation (12)) for the circulating and fixed capital cases, respectively. It is thus ascertained that the linear approximation works pretty well – especially – in the fixed capital case, where the second eigenvalue of **J** is approximately equal to –0,099 and, in general, the eigenvalue decay is remarkably fast.

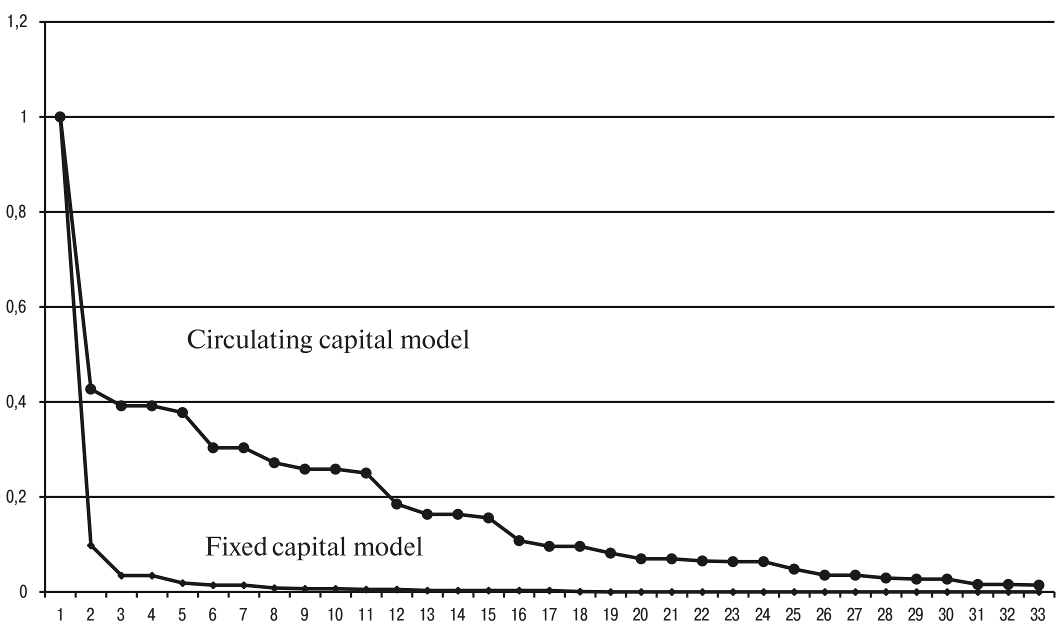
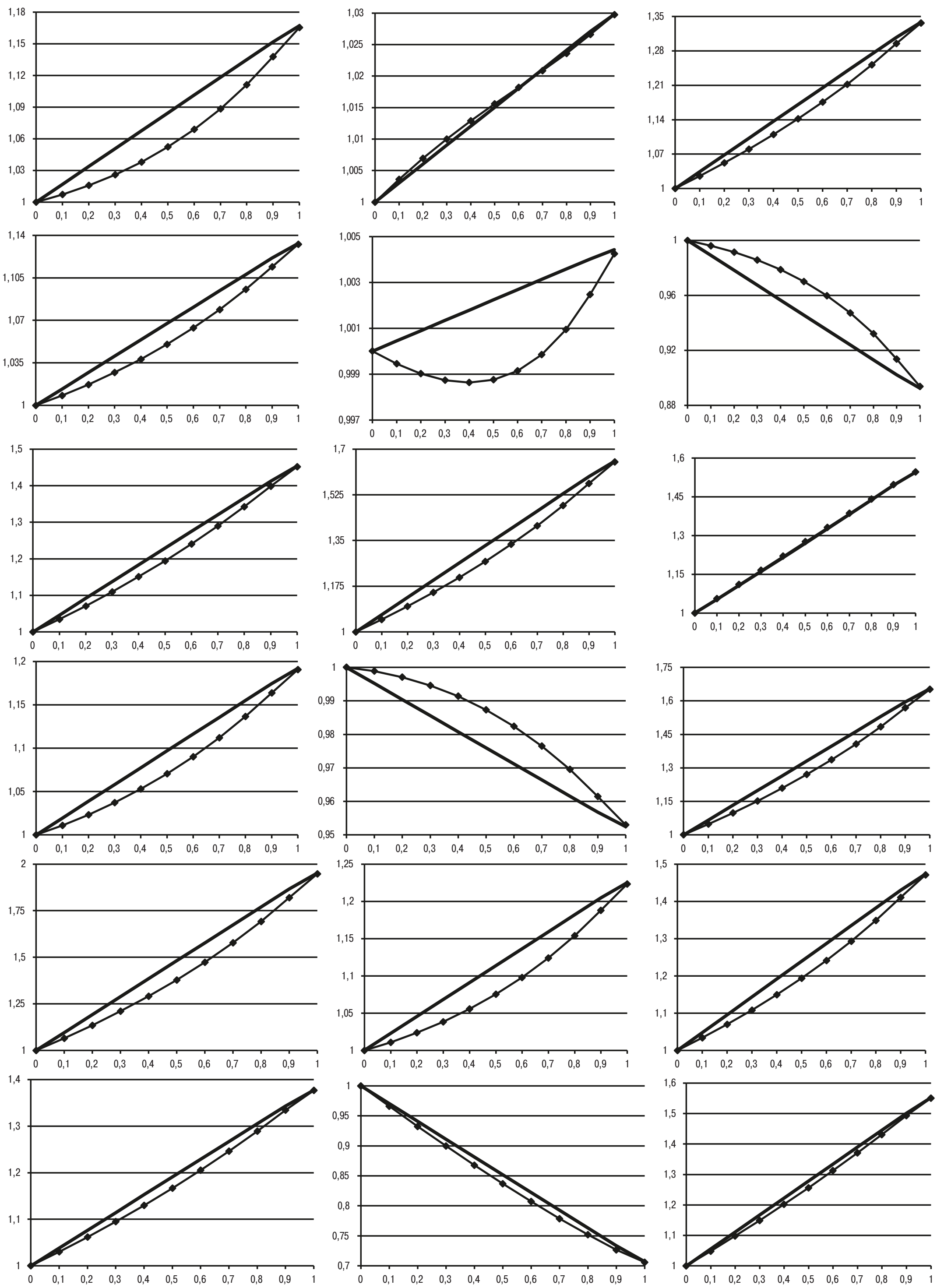


Fig. 8. The moduli of the eigenvalues; circulating and fixed capital models

At first sight the graphs in Fig. 9 and 10 may give the impression of large deviations in the middle range of the relative rate of profit. However, on closer examination the deviations are surprisingly small given the parsimony in the terms used for the approximation. In effect, in the case of circulating capital model the maximum absolute deviation is 9,64% and typically the average deviation (excluding the values for $\rho = 0$ and $\rho = 1$) is 2,06%. Similarly, in the case of fixed capital stock model the maximum deviation is 3,36% and the average deviation (excluding the extreme values of ρ) is 0,88%.



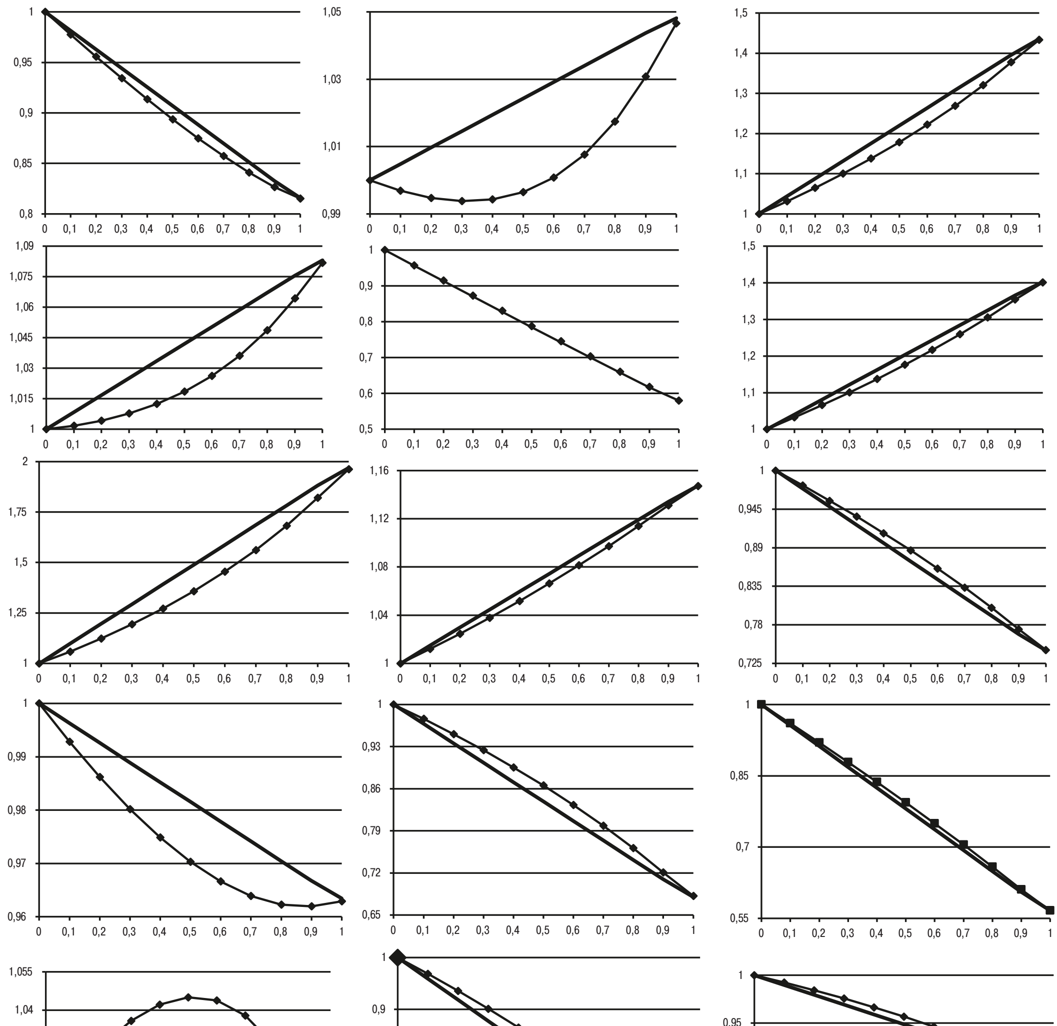
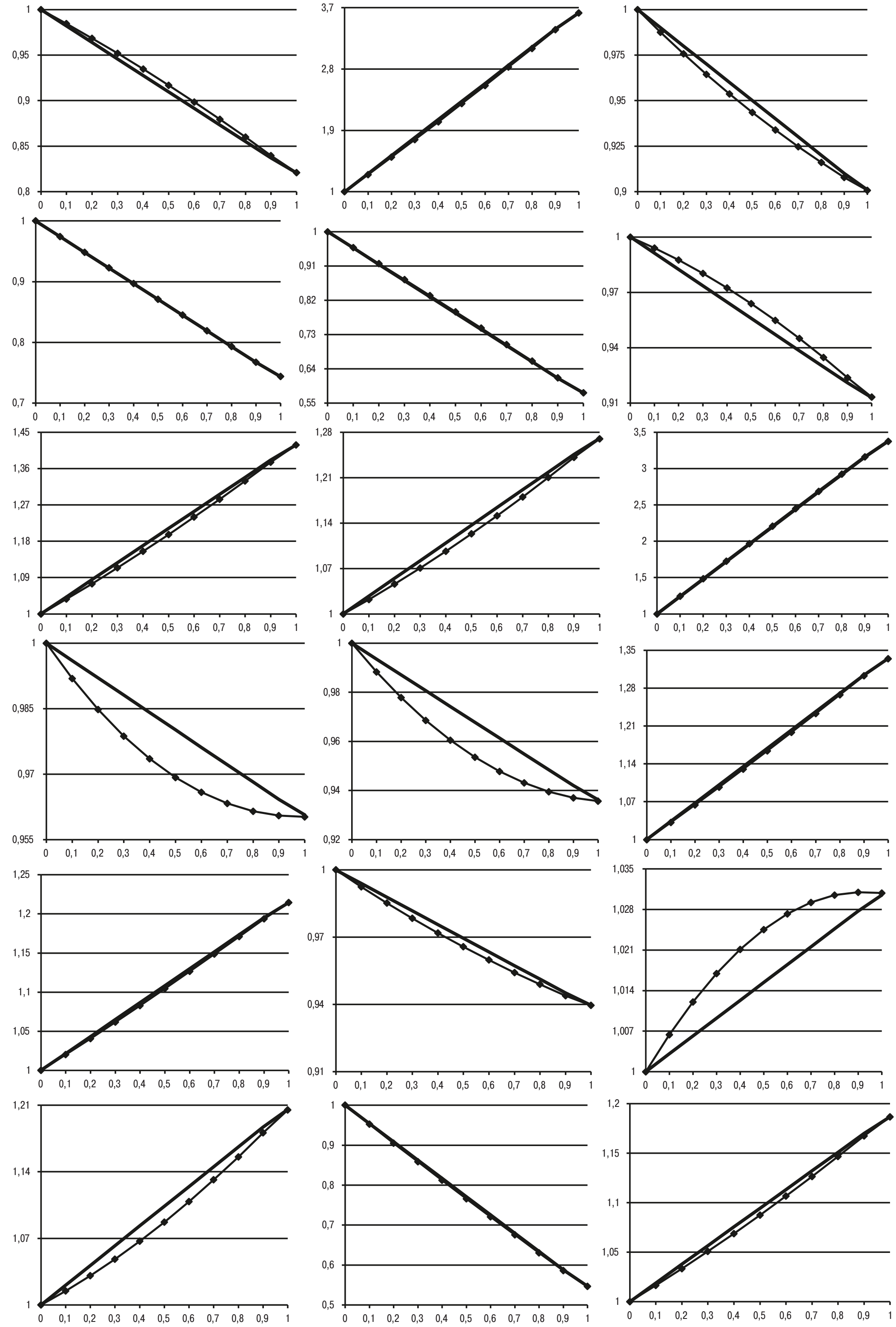


Fig. 9. Actual and approximate price curves; circulating capital model

The deviations between the approximation identified with the straight line and the actual trajectory which includes all the terms are surprisingly small given the parsimony in the terms used for the approximation. In effect, the maximum absolute deviation is 9,64% and typically the average absolute deviation (excluding the values for $\rho = 0$, $\rho = 1$) is 2,06%. This means that the inclusion of more terms does not add more to our knowledge of the actual movement of the production price relative rate of profit curves provided that the maximum deviations occur at unusually high relative rates of profit.


1 0,95 0,9 0,85 0,8
3,7 2,8 1,9 1
1 0,975 0,95 0,925 0,9
1 0,9 0,8 0,7
1 0,91 0,82 0,73 0,64 0,55
1 0,97 0,94 0,91
1,45 1,36 1,27 1,18 1,09 1
1,28 1,21 1,14 1,07 1
3,5 3 2,5 2 1,5 1
1 0,985 0,97 0,955
1 0,98 0,96 0,94 0,92
1,35 1,28 1,21 1,14 1,07 1
1,25 1,2 1,15 1,1 1,05 1
1 0,97 0,94 0,91
1,035 1,028 1,021 1,014 1,007 1
1,21 1,14 1,07 1
1 0,9 0,8 0,7 0,6 0,5
1,2 1,15 1,1 1,05 1
0 0,1 0,2 0,3 0,4 0,5 0,6 0,7 0,8 0,9 1

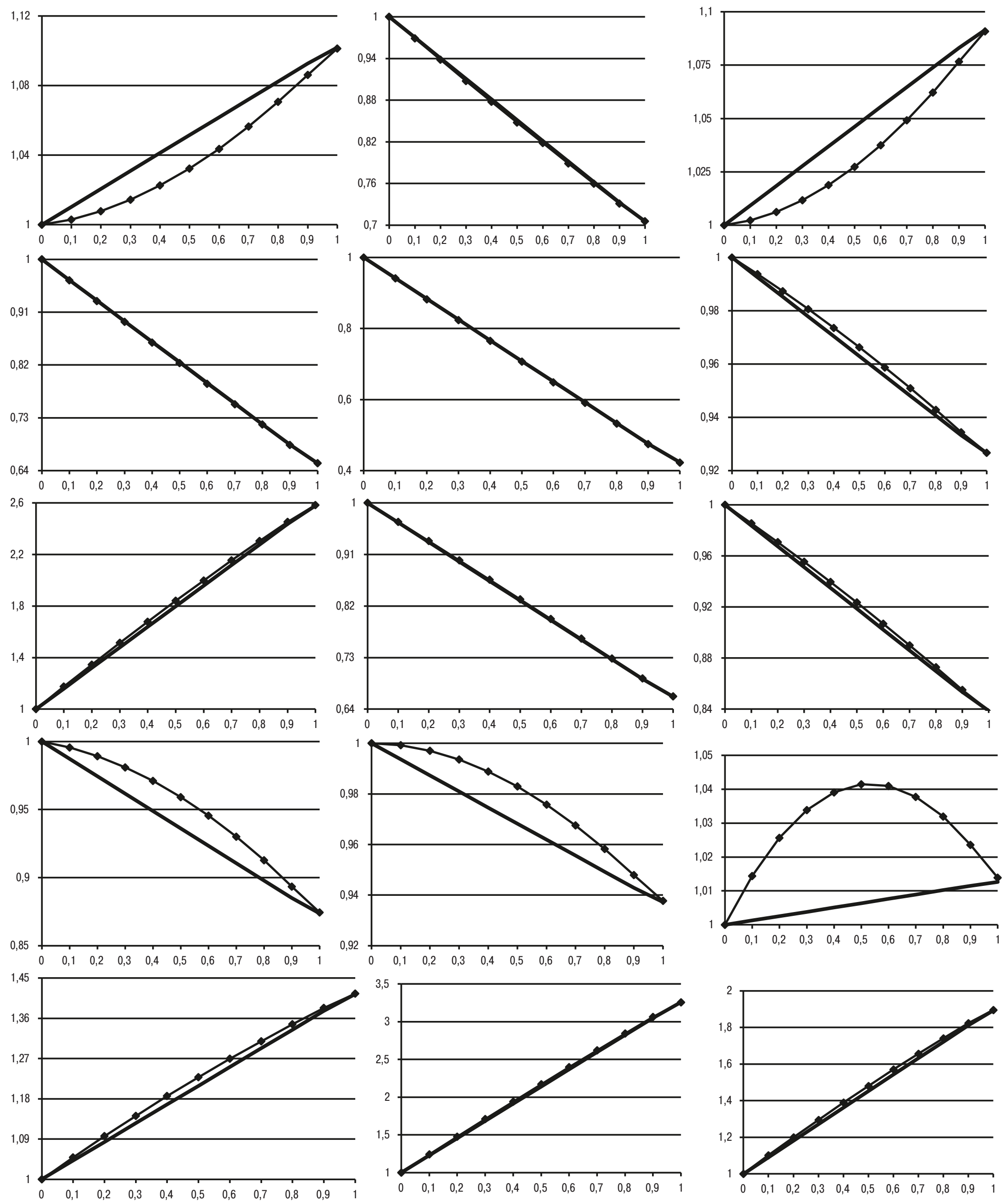


Fig. 10. Actual and approximate price curves; fixed capital model

We observe that the deviations in the case of fixed capital stock model are much smaller than those in the circulating capital. The maximum absolute deviation is only 3,36% and typically the average deviation (excluding the extreme values for $\rho = 0$, $\rho = 1$) is 0,88%. It is interesting to reiterate that the maximum deviation occurs at relative rates of profit of 50 or 60% which happen to be unusually high given that the realistic value of the relative rate of profit is in the area of 25 to 30%.

5. The Hyper-Basic Industry

Having established that $J^A \equiv (y_{J1} x_{J1}^T)^{-1} x_{J1}^T y_{J1}$ is a pretty good approximation of **J**, in the sense that both matrices give rise to price trajectories quite similar to each other, we shall focus on the fixed capital case and apply the transformations (11)-(11a) to these matrices.м

The two matrices (i) and (ii) below are derived by means of a Schur triangularization of **J**$^{\mathbf{A}}$ and the original **J**, respectively. It then follows that:

(i). The first row of the semi-positive and rank-1 matrix $\tilde{J}^A \equiv T^{-1} J^A T$ is

1,000000	0,792316	0,355025	0,301896	0,297024	0,362482	0,463247
0,384227	0,653548	0,370216	0,356369	0,349827	0,320841	0,344878
0,304707	0,462918	0,346286	0,347760	0,535457	0,330577	0,431858
0,226715	0,292348	0,283885	0,673378	0,279085	0,344606	0,28279
0,373765	0,345979	0,455069	1,234081	0,419608		

(ii). The first row of the non-semipositive and rank-30 matrix $\tilde{J} \equiv T^{-1} J T$ is

1,000000	0,790728	0,336137	0,276442	0,298501	0,375249	0,366382
0,301568	0,554690	0,308661	0,286090	0,302050	0,282940	0,295594
0,294691	0,358277	0,293474	0,318139	0,346457	0,279794	0,337487
0,208219	0,281176	0,264605	0,715178	0,289506	0,381651	0,390657
0,424787	0,204948	0,550097	1,574976	0,694034		

(iii). The Euclidean norm of the difference between these rows is almost 0,585, and the spectral norm of $\tilde{J} - \tilde{J}^A$ is almost 0,760.

Thus, it is concluded that the transformed approximate matrix $\tilde{\mathbf{J}}^{\mathrm{A}}$ extracts the essential information embedded in the original system and that, in the fixed capital case, even the original system tends to be economically equivalent to an $n \times n$ 'corn-tractor' system. For the circulating capital case, this statement is ascertained only in a weaker sense.

6. Concluding Remarks

This paper has shown that the properties of an economy with regard to the trajectories of prices in the face of changes in the relative profit rate are regulated to a great extent by the configuration of the eigenvalues contained in its input-output structure. The weight of hitherto evidence suggests that the distribution of eigenvalues typically follows a rectangular hyperbola-like scheme and the second eigenvalue is usually much smaller than the first eigenvalue and the difference between the first and the second eigenvalue increases markedly, when we include the matrix of capital stock. This characteristic distribution of eigenvalues implies that the quasi-linearity of the wage-price-profit rate relationships. In this paper, we ascertained these findings in the case of the UK economy, which we examined exclusively for illustrative purposes. In addition, we made a further step showing that for the fundamental properties of the actual economies, we need not know or use all the available information, but rather that the structure of the complex economy can be disentangled to a much simpler and parsimonious structure which includes an (or some) hyper-basic industry (industries). This characteristic industry can be extracted by means of the Schur triangularization theorem (or more general decompositions), and it is interesting to note that the hyper-basic industry bears similarities with Sraffa's Standard industry in the sense that both are hypothetical and both derive their important properties from the technological characteristics of actual economies.

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Appendix 1

In what follows, we provide the sources of the data and the methods followed for the computation of the various matrices and vectors for the UK economy based on OECD STAN publications (http:/www.oecd.org). The matrix of input-output coefficients, A, for the year 1990 is obtained by dividing element-by-element the inputs of each industry by its gross output, x all given in millions of UK pounds.

The vector of direct labour coefficients l is estimated by dividing the industry employment by the respective gross output and the result is multiplied by the ratio of the average wage of each industry over the economy-wide minimum industry wage in an effort to account for the probable interindustry differences in skills. Fortunately, the total wages for each of our 33 industries are given in the input-output table, which are presumably the product of the average wage rate in the industry times the number of employees. In order to obtain the total employment of each sector we need to estimate the wage equivalent of the self-employed population. In order to do that, we divide the total employment (employees plus self-employed) by the number of employees and then we multiply by each industry's average wage.

The matrix of capital stock, that is, the production and allocation of capital goods between industries per unit of output produced is not directly available in the input-output tables in general it can be constructed though indirectly through the use of capital (investment) flows table which allocates the investment flows of each industry to itself and others. It is worth pointing out that such tables are published sporadically, but in the recent years when published are no longer in their convenient for us symmetric form, as we are given the various investment goods allocated to various industries and from the available data is extremely difficult to construct a symmetric investment flow matrix. Fortunately, the OECD STAN data base contains such flow matrices for the UK, USA and Japan for the year 1990 and from these flow matrices the one for the UK (1990) fits very well with the symmetric input-output table and the capital stock vector for the same year.

For the construction of the capital stock matrix we use the investment flows matrix to form weights, assuming in the absence of an actual capital stock matrix that the capital stock is allocated among producing industries in a way similar to that of investment. The matrix of weights is in turn is multiplied element-by-element by the vector of net capital stock estimates for each industry published in OECD STAN data base. The matrix that we got, we applied our usual division by the output of each industry. The matrix of depreciation coefficients could be constructed in similar fashion however the industry data for depreciation were not readily available and furthermore the depreciation we know from a number of studies strengthen the results of monotonicity of wage-production price-profit rate curves.

Appendix 2

Industry Nomenclature

1. Agriculture, forestry & fishing
2. Mining & quarrying
3. Food, beverages & tobacco
4. Textiles, apparel & leather
5. Wood products & furniture
6. Paper, paper products & printing
7. Industrial chemicals
8. Drugs & medicines
9. Petroleum & coal products
10. Rubber & plastic products
11. Non-metallic mineral products
12. Iron & steel
13. Non-ferrous metals
14. Metal products
15. Non-electrical machinery
16. Office & computing machinery
17. Electrical apparatus, nec
18. Radio, TV & communication equipment
19. Shipbuilding & repairing
20. Other transport
21. Motor vehicles
22. Aircraft
23. Professional goods
24. Other manufacturing
25. Electricity, gas & water
26. Construction
27. Wholesale & retail trade
28. Restaurants & hotels
29. Transport & storage
30. Communication
31. Finance & insurance
32. Real estate & business services
33. Community, social & personal services

Appendix 3

List of Eigenvalues

Circulating Capital	Fixed Capital Stock
1	1
0,42729999	–0,098654032
0,39161693 + 0,0094387002i	0,024738469 + 0,024187749i
0,39161693 – 0,0094387002i	0,024738469 – 0,024187749i
0,37733932	0,01898522
0,30101897 + 0,038174116i	–0,0082134922 + 0,011905663i
0,30101897 – 0,038174116i	–0,0082134922 – 0,011905663i
0,25806034 + 0,013605372i	0,008828955
0,25806034 – 0,013605372i	0,0037200892 + 0,0056788125i
0,27215789	0,0037200892 – 0,0056788125i
0,25014899	–0,00058736565 + 0,0055590160i
0,18512913	–0,00058736565 – 0,0055590160i
0,15448346 + 0,054431503i	–0,00039696986 + 0,0029355917i
0,15448346 – 0,054431503i	–0,00039696986 – 0,0029355917i
0,013012591 + 0,068882007i	–0,0021687989 + 0,0018838427i
0,013012591 – 0,068882007i	–0,0021687989 – 0,0018838427i
0,15614117	0,003293596
–0,014666727	0,001126079
0,0061682271 + 0,026133636i	–0,000480193
0,0061682271 – 0,026133636i	0,00035192
0,10836522	–2,0204005e–006 + 0,00020717917i
0,093683094 + 0,021977362i	–2,0204005e–006 – 0,00020717917i
0,093683094 – 0,021977362i	–8,34409E–05
0,015591854 + 0,0013782113i	8,19863E–05
0,015591854 – 0,0013782113i	–3,6672858e–005 + 1,7877046e–005i
0,033203803 + 0,012221087i	–3,6672858e–005 – 1,7877046e–005i
0,033203803 – 0,012221087i	3,5233736e–005 + 3,1383270e–005i
0,062252619 + 0,012647012i	3,5233736e–005 – 3,1383270e–005i
0,062252619 – 0,012647012i	1,48145E–05
0,047795835	6,64E–07
0,065586197	0
0,029401245	0
0,082013049	0